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STRESS STATE AROUND CYLINDRICAL CAVITIES IN ISOTROPIC MEDIUM

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ABSTRACT

The present paper is dealing with the investigation of the stress field around the infinitely long cylindrical cavity, of circular cross section, contained in elastic continuum. Investigation is based upon the determination of the stress function that satisfies the biharmonic equation, for the given boundary conditions and for rotationaly symmetrical loading. The solution of the partial differential equation of the problem is given in the form of infinite series of Bessel functions. The particular contribution of the present paper to already existing investigations of the considered problem is the introduction and definition of the new functions representing the loading, which is treated as the corresponding stress boundary conditions. As the verification and illustration of obtained analytical solutions, the paper also contains numerical example.

Key words**:** *cylindrical cavity, stress state, rotationally symmetrical loading, functions of loading*

STANJE NAPONA OKO CILINDRIČNIH ŠUPLJINA U IZOTROPNOJ SREDINI

REZIME

U ovom radu se istražuje stanje napona oko beskrajno duge cilindrične šupljine, kružnog poprečnog preseka, koja se nalazi u elastičnom kontinuumu. Istraživanje se zasniva na određivanju naponske funkcije koja zadovoljava biharmonijsku jednačinu, za navedeni granični uslov i za rotaciono simetrično opterećenje. Rešenje parcijalne diferencijalne jednačine ovog problema je dato u obliku beskonačnog reda Beselovih funkcija. Poseban doprinos ovog rada je da se postojeća istraživanja unaprede, a problemi definisanja opterećenja prevaziđu uvođenjem i definisanjem novih funkcija koje predstavljaju opterećenje, a koje daju odgovarajuće naponske granične uslove. Kao verifikacija i ilustracija dobijenih analitičkih rešenja, rad takođe sadrži numeričke primer.

Ključne reči: cilindrična šupljina, naponsko stanje, rotaciono simetrično opterećenje, funkcije opterećenja

INTRODUCTION

Investigation of the stress state around the cavity in the shape of an infinitely long circular cylinder has been the subject of interest even in the early work of [3], where the determination of the stress state has been treated as the plane problem. Evaluation of the stress state around the infinitely long cylindrical cavity contained in isotropic and homogeneous elastic medium, based on determination of the stress functions, may be found in work of many authors [1],[2],[5],[6],[7],[8], etc. The review may be seen in [4].

The present paper is concerned with the investigation of the stress field around the infinitely long cylindrical cavity that is loaded or partially loaded at the inner surface by the rotationally symmetric loading. As the contribution to the similar investigations, the paper introduces the new function of loading in the form of the infinite sine series. The previously elaborated loading functions, developed in infinite series on the basis of Legendre's polynomials [4] or defined by Fourier series, Jaeger and Cook, have had the inherent shortcoming at the edges of loaded surface, that is avoided in this paper.

EXPRESSION FOR STRESSES

Determination of the state of stress around the opening in the shape of infinitely long circular cylinder, for the case of rotationally symmetric loading, is reduced to determination of the stress function Ψ which satisfies the biharmonic equation, [2]:

$$\nabla^2\nabla^2\Psi = 0 \tag{1}$$

If the cylindrical coordinate system is used, stresses are related to the stress function in the following form:

$$\sigma_z^* = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Psi - \frac{\partial^2\Psi}{\partial z^2}\right]$$

$$\sigma_\varphi^* = \frac{\partial}{\partial z}\left(\nu\nabla^2\Psi - \frac{1}{r}\frac{\partial\Psi}{\partial r}\right)$$

$$\tau_{rz}^* = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Psi - \frac{\partial^2\Psi}{\partial z^2}\right] \tag{2}$$

where the operator ∇^2, for the case of rotational symmetry, is defined by

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{\partial\Psi}{r\partial r} + \frac{\partial^2\Psi}{\partial z^2} \tag{3}$$

while ν represents the Poisson's coefficient. Eq. (1) therefore, obtains the following form:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]\left[\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial z^2}\right] = 0 \tag{4}$$

The solution of differential equation (4) is is assumed as

$$\Psi = \sum_{k=0}^{\infty} f(kr)\sin kz \tag{5}$$

Substituting expression (5) into (4) one obtains

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\mathrm{d}}{r\,\mathrm{d}r}-k^2\right)\left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{\mathrm{d}f}{r\,\mathrm{d}r}-k^2 f\right)=0 \tag{6}$$

where the differential operator (3) has now the form:

$$\left(\overline{\nabla}^2\right)\equiv\frac{d^2}{dr^2}+\frac{d}{rdr}-k^2 \tag{7}$$

Applying the operator (7) to the function $f(kr)$, one obtains the ordinary Bessel's differential equation [9] of the second kind, for the case of $n=0$:

$$\frac{d^2 f}{dr^2}+\frac{df}{rdr}-k^2 f=0 \tag{8}$$

The general solution of differential Eq. (8) is given by:

$$f(kr)=A_1 J_0(kr)+B_1 K_0(kr) \tag{9}$$

where $J_0(kr)$ and $K_0(kr)$ are the Bessel functions of order zero, while A_1 and B_1 are the corresponding constants of integration.

Applying the recurrent formulae and relations between the Bessel's functions and their first derivatives, one could derive expressions for stresses according to Eqs. (2). Obtained stress expressions contain the Bessel's functions of order zero and one. For the limiting case of $r\to\infty$ one obtains the solution in the form:

$$f(kr)=B_1 K_0(kr)+B_2 rK_1(kr) \tag{10}$$

where $K_1(kr)$ is the Bessel's function of order one, while B_2 is the corresponding constant of integration.

Therefore, inserting Eq. (10) into Eq. (5) the solution of differential Eq. (4) is obtained as

$$\Psi(r,z)=\sum_{k=0}^{\infty}[B_1 K_0(kr)+B_2 rK_1(kr)]\sin kz \tag{11}$$

Consequently, expressions Eqs. (2) for additional stresses due to the presence of cylindrical cavity are obtained as:

$$\sigma_r^*=\sum_{k=0}^{\infty}k^2\cos kz\left\{K_0(kr)\left[B_2(1-2\nu)-kB_1\right]-K_1(kr)\left(krB_2+\frac{B_1}{r}\right)\right\}$$

$$\sigma_\varphi^*=\sum_{k=0}^{\infty}k^2\cos kz\left[B_2(1-2\nu)K_0(kr)+\frac{B_1}{r}K_1(kr)\right]$$

$$\sigma_z^*=\sum_{k=0}^{\infty}k^2\cos kz\left\{K_0(kr)\left[kB_1-2(2-\nu)B_2\right]+B_2krK_1(kr)\right\}$$

$$\tau_{rz}^{*} = \sum_{k=0}^{\infty} k^{2} \sin kz \left\{ -B_{2} kr K_{0}\left(kr\right) - [kB_{1} - 2\left(1-\nu\right) B_{2}] + K_{1}\left(kr\right) \right\} \tag{12}$$

BOUNDARY CONDITIONS

Boundary conditions required for determination unknown constants B_1 and B_2 in expressions for stresses (12) are given in terms of stresses acting on the cavity surface. The total stresses around the cavity are determined by superimposing primary stresses acting in the continuum without a cavity and "partial" stresses which are caused by the presence of cavity.

$$\sigma_{i} = \sigma_{i}^{*} + \sigma_{i}^{pr} \ ; \qquad \tau_{ij} = \tau_{ij}^{*} + \tau_{ij}^{pr} \tag{13}$$

where:

σ_i , τ_{ij} = the total stresses;

σ_i^* , τ_{ij}^* = partial stresses due to the presence of a cavity;

σ_i^{pr} , τ_{ij}^{pr} = primary stresses;
(i, j = respective coordinate)

In the representation of the boundary conditions the hidrostatic model of rock medium is adopted. In that case one has the primary stresses given as

$$\sigma_{r}^{pr} = \sigma_{\varphi}^{pr} = \sigma_{z}^{pr} = \gamma H \ ; \qquad \tau_{r\varphi}^{pr} = \tau_{rz}^{pr} = \tau_{z\varphi}^{pr} = 0 \tag{14}$$

where γ represents the unit weight of the rock and H the vertical coordinate of the considered point. At the internal side of the surface of the cylindrical cavity, the boundary conditions are given in the form

$$\left. \begin{aligned} \sigma_{r}^{*} &= -\sigma_{r}^{pr} - p\left(\frac{z}{R}\right) \\ \tau_{rz}^{*} &= -\tau_{rz}^{pr} = 0 \end{aligned} \right\} \quad \text{for} \quad r = R \tag{15}$$

The state of stress around the cylindrical cavity loaded by partialy distributed rotationaly symmetric loadind over the interior side, see Fig. 1, is considered in work of several authors.

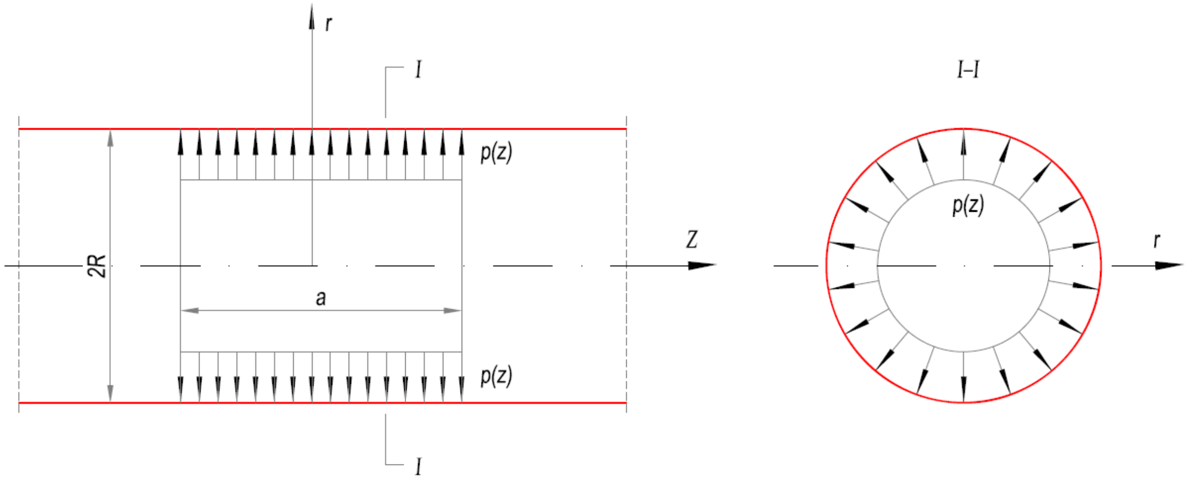

Figure 1 Partially distributed rotationaly symmetric loading
Slika 1.Parcijalno rotaciono simetrično opterećenje

The main task is the definition of the corresponding rotationaly symmetric loading. In case of a partially loaded cavity, within the limits: -B/2 ≤ z ≤ B/2 or 0 ≤ z ≤a , the loading function has been defined [2], in the following form

$$p\left(\frac{z}{r}\right)=\frac{2p}{\pi}\int_0^\infty \frac{1}{\alpha_1}\sin\frac{\alpha_1 B}{2r}\cos\frac{\alpha_1 z}{r}\,d\alpha_1 \tag{16}$$

where p is constant loading. By introducing in Eq. (16) the substitution: $x = \alpha_1 B/2r$ and after the mathematical transforms one can obtain the loading function as

$$p\left(\frac{z}{r}\right)=\frac{2p}{\pi}\int_0^\infty \frac{\sin x\cos Ax}{x}dx \tag{17}$$

where: $A = 2z/B$. The integral Eq. (17) is the well known in the literature and the loading can be obtained from Eq. (16) and Eq. (17) as

$$\begin{aligned} &p(z/r)=p && \text{for} \quad z<|B/2|\,; \\ &p(z/r)=p/2 && \text{for} \quad z=|B/2| \\ &p(z/r)=0 && \text{for} \quad z>|B/2| \end{aligned} \tag{18}$$

From the relationships (18) one can conclude that the loading function has the constant value inside the interval ($-B/2$, $B/2$) , but at the end points $-B/2$, and $B/2$ (i.e. 0 and a) it has the discontinuity of the first kind, and the final loading value of $p/2$. Outside of these points the value of the function is zero. This is the reason to notify so far unresolved problem of the existence of the solution of the basic equation $\nabla^2\ \nabla^2 = 0$ in the vicinity of these end points. However, in the present paper the partially distributed rotationaly symmetric loading is represented by the infinite series in the form, Fif. 2:

$$p(z)=p\sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\frac{(2n+1)\pi}{a}(a-z);\quad 0\le z\le a$$

$$p(z)=0;\qquad\qquad z\le 0\ \wedge\ z\ge a \tag{19}$$

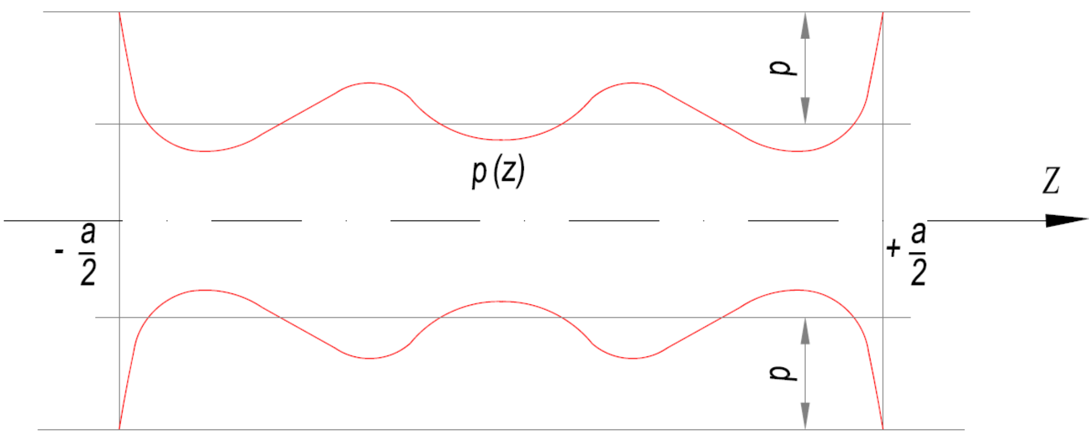

Figure 2 Graphical representation of the loading
Slika 2. Grafički prikaz opterećenja

The selection of loading functions in the cited problem may be considered from different aspects, depending on the aims. In this presentation, the comparative study has been made with the aim to compare the previously used loading function and the recently introduced ones. The comparison is helped by the graphical presentations given in the Fig.3.

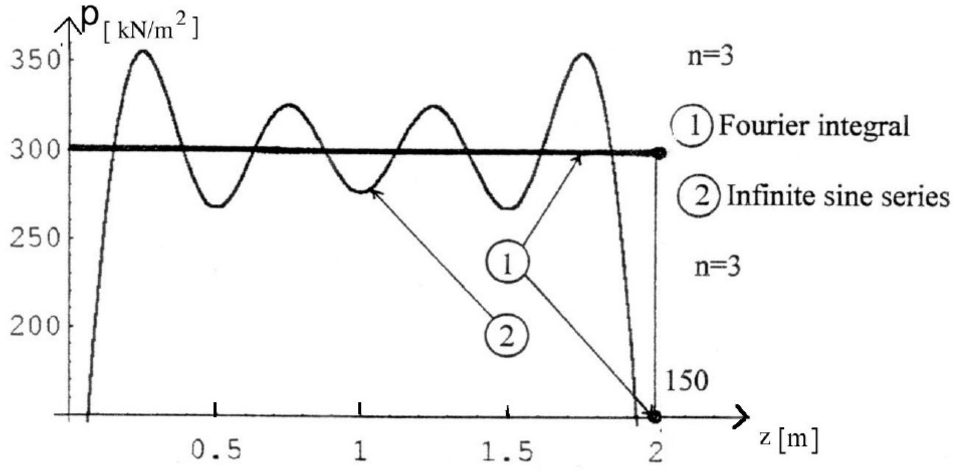

Figure 3 Comparative analysis of functions (p = 300 kN/m , a =2 m)
Slika 3. Uporedna analiza funkcija (p = 300 kN/m , a =2 m)

By comparing the graphs of functions based on Fourier integrals, (see Fig. 3), which has the discontinuity of the first kind at interval ends, to the loading function based on sine series, which is continuous, one may conclude that it is always more prudent from the mathematical considerations to use the continuos function over all loaded area and its ends.

For partially distributed loading defined by Eq. (19), unknown integration constants B_1 and B_2 are determined from the given boundary conditions (15), in the following form:

$$B_1 = B_2[\frac{2(1\text{-}\nu)}{k} - r\frac{K_0(kr)}{K_1(kr)}]$$

$$B_2 = -\frac{\sigma_r^{pr} + p\sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\frac{(2n+1)\pi}{a}(a-z)}{\sum_{k=0}^{\infty}k^2\cos kz(kR\frac{K_0^2(kr)}{K_1(kr)} - K_1(kr)[\frac{2}{kR}(1\text{-}\nu)+kR])} \qquad (20)$$

When the constants of integration B_1 and B_2 are determined, the stresses are obtained from expressions (12) and (13).

NUMERICAL EXAMPLES

In order to analyze obtained analytical solutions, the parametric calculation of stresses is performed by using the computer program Mathematica. The following numerical values defining the rock, the cavity and the loading over the internal boundary (r.e. the boundary conditions) are addopted

- Parameters of the rock:
  $\gamma = 28$ kN/m^3
  $\nu = 0.3$
  $E = 20 \times 10^6$ kN/m^2
- Geometrical parameters:
  $H = 100\,m$
  $r_0 = 1.5\,m,\ \ r_0 = 2.0\,m,\ \ r_0 = 2.5\,m$
  $\Delta r = 0.2\,m$

- Boundary conditions:
  p = 300 kN/m^2
  a = 2 m

Obtained results are presented in the graphical form in Figs. 4 and 5.

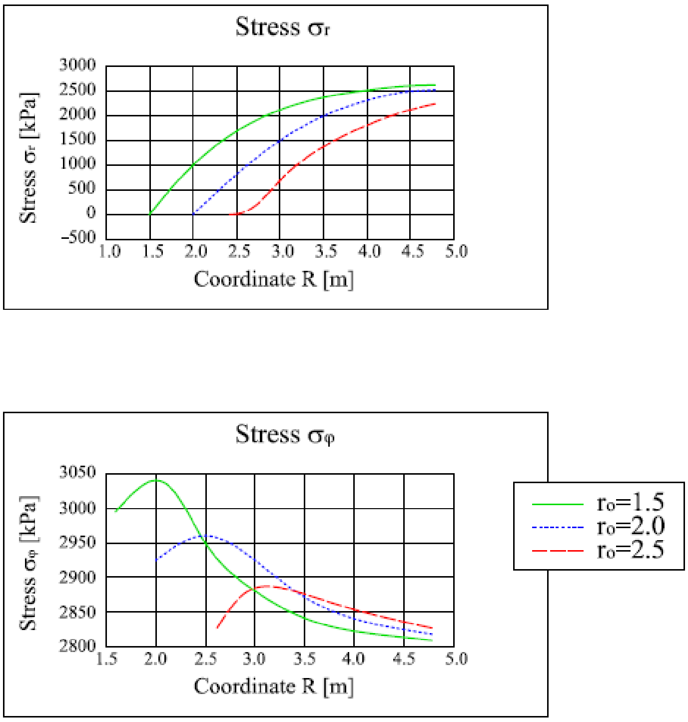

Figure 4 Numerical results: free boundary
Slika 4. Numerički rezultati: slobodna kontura

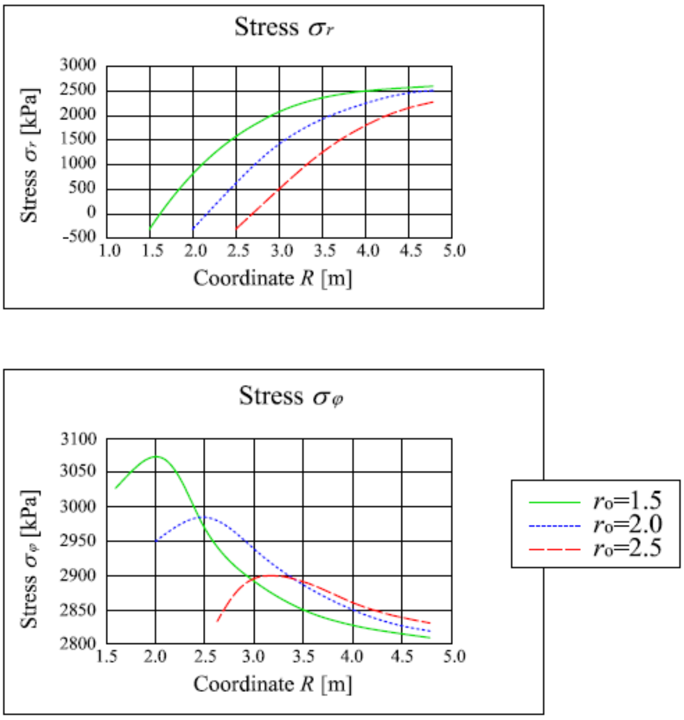

Figure 5 Numerical results: loaded boundary
Slika 5. Numerički rezultati: opterećena kontura

Presented numerical results enable better insight into the problem of the stress field around cylindrical cavities. Particularly useful this analysis may be for the analysis of either unsupported or partially supported excavation in rock masses and the influence of the supporting structure on the state of stresses in the rock mass near the cavity. In such a way it is possible to establish the optimum loading that could represent the supporting structure in excavation.

On the other hand, analytically obtained stress diagrams for partially loaded cavity could be compared with stress values obtained by measurement of working stresses, in order to interpret the measured values.

CONCLUSIONS

The paper is presenting the possibility to use analytically obtained solutions to evaluate the state of stress to some practical problems, especially in construction of underground objects. The new original boundary conditions has been introduced, using series of a single parameter (sine) function that provided improved convergency compared to the previously used loading functions.

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